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A New Class of Modified Bernstein Operators

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The left Bernstein quasi-interpolant operator introduced by Sablonnière is a kind of modified Bernstein operator that has good stability and convergence rate properties. However, we recently found that it is not very convenient for practical applications. Fortunately, we showed in a previous paper that there exist many operators having stability and convergence rate properties similar to those of Sablonnière's operator. In this paper, we introduce another specific class of such operators generated from the operator introduced by Stancu. We present detailed results about this class, some of which can be applied to numerical quadrature. Finally, we clarify its advantages and assert that it is more natural and more convenient, both theoretically and practically, than that of Sablonnière. Our paper, at the same time, provides several new results regarding Stancu's operator.

1. INTRODUCTION

The left Bernstein quasi-interpolant operator $B_n^{(K)}$ $(n \in \mathbf{N},\ K \in \mathbf{N}_0 \overset{\text{def}}{=} \mathbf{N} \cup \{0\})$ was introduced by Sablonnière in [6, 7] and redefined by us in [3] as

$$B_n^{(K)} f = \sum_{k=0}^{\{K,n\}} U_k^n (B_n f)^{[k]} \qquad (f\colon [0,1] \to \mathbf{R}),$$

where B_n is the Bernstein operator of order n and U_k^n are the unique polynomials satisfying

$$L_n f = \sum_{k=0}^{n} U_k^n (B_n f)^{[k]} \qquad (f\colon [0,1] \to \mathbf{R}), \tag{1.1}$$

where L_n is the Lagrange operator of the same nodes as B_n. (The symbol $\sum_{k=0}^{\{K,n\}}$ stands for $\sum_{k=0}^{\min\{K,n\}}$ and $(B_n f)^{[k]}$ stands for $(B_n f)^{(k)}/k!$. We use the notations like these throughout this paper.) It was shown in [7] that

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$B_n^{(K)}f - f = O(n^{-([K/2]+1)})$ (pointwise) for every f sufficiently smooth, while the boundedness of $\{\|B_n^{(K)}\|\}_n$ was guaranteed in [11], where $\|\cdot\|$ is the operator norm subordinate to the uniform norm on $C[0, 1]$. Furthermore, we generalized and refined these results in [3].

However, $B_n^{(K)}$ is not very convenient for practical applications. The sequence $\{B_n^{(K)}\}_n$ is indeed stable in the sense that it is uniformly bounded, but the value of $\|B_n^{(K)}\|$ grows extremely fast as K increases, especially when $n(\geqslant K)$ is near to K. In fact, when $n = K$, the operator $B_n^{(K)}$ reduces to L_K, whose norm grows exponentially with respect to K, as is well known. Table I is a result of numerical experiments (this kind of table was given also in [6]), where we omit the values in the case $n < K$ because they are not defined originally, and according to our new definition, $B_n^{(K)}$ trivially reduces to $B_n^{(n)}(=L_n)$.

There is another defect regarding $B_n^{(K)}$. Though it was introduced for the purpose of accelerating convergence of Bernstein polynomials, the correspondence between the parameter K and the order of its convergence rate is not one-to-one. In fact, for every nonnegative integer α, the convergence rates of $\{B_n^{(2\alpha)}f\}_n$ and $\{B_n^{(2\alpha+1)}f\}_n$ are both $O(n^{-(\alpha+1)})$. The operator $B_n^{(K)}$ was constructed by truncating at $k = K$ the expansion (1.1) of L_n, but the above fact suggests that the mode of truncation is not essential for our original purpose.

Fortunately, we showed in [3] that there exist many operators having stability and convergence rate properties similar to those of Sablonnière's operator. In this paper, we will introduce another specific class of such operators. The new operator ${}_\alpha B_n$ ($n \in \mathbf{N}$, $\alpha \in \mathbf{N}_0$) is generated from the

TABLE I

Approximate Values of $\|B_n^{(K)}\|$

	K							
n	2	3	4	5	6	7	8	16
2	1.250	—	—	—	—	—	—	—
3	1.250	1.631	—	—	—	—	—	—
4	1.250	1.529	2.208	—	—	—	—	—
5	1.204	1.482	2.100	3.106	—	—	—	—
6	1.196	1.454	2.038	2.980	4.549	—	—	—
7	1.189	1.436	1.997	2.899	4.389	6.930	—	—
8	1.180	1.424	1.969	2.843	4.279	6.714	10.946	—
9	1.175	1.415	1.948	2.801	4.199	6.557	10.639	—
10	1.172	1.407	1.932	2.769	4.137	6.438	10.408	—
16	1.158	1.385	1.887	2.676	3.951	6.078	9.713	934.534
32	1.150	1.372	1.854	2.607	3.817	5.820	9.218	832.241

operator introduced by Stancu [8], and also can be regarded as a truncated operator of L_n, but the mode of truncation is distinct from that of the preceding operator. We will present some detailed results about our new operator and several propositions about Stancu's operator. Finally, we will compare it with Sablonnière's operator, clarifying advantages of our new operator.

2. MAIN RESULTS

2.1. *Definition*

Stancu [8] introduced the operator $P_n^{\langle s\rangle}$ for every $n\in\mathbf{N}$ and for every $s\in\mathbf{R}$ satisfying $\prod_{\mu=0}^{n-1}(1+\mu s)\neq 0$, as

$$P_n^{\langle s\rangle}f(x)=\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu}\frac{[\prod_{\mu=0}^{\nu-1}(x+\mu s)][\prod_{\mu=0}^{n-\nu-1}(1-x+\mu s)]}{\prod_{\mu=0}^{n-1}(1+\mu s)}$$

$$(f\colon[0,1]\to\mathbf{R},\ x\in[0,1]).$$

(We adopt the convention that $\sum_p^q\cdot=0$, $\prod_p^q\cdot=1$ if $q<p$.) This operator has the two identities

$$P_n^{\langle 0\rangle}f(x)=\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}=B_nf(x),$$

$$P_n^{\langle -1/n\rangle}f(x)=\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{nx}{\nu}\binom{n(1-x)}{n-\nu}=L_nf(x),$$

which mean that the class of Stancu operators contains the Bernstein and the Lagrange ones. Stancu investigated in particular the case $s\geqslant 0$ as a class of positive linear operators. However, here we treat Stancu's operator from quite a different standpoint. We use it to introduce a new class of operators as follows.

DEFINITION 2.1. We define the modified Bernstein operator ${}_{\alpha}B_n$ (of order $n\in\mathbf{N}$ and sharpness degree $\alpha\in\mathbf{N}_0\cup\{\infty\}$) as

$${}_{\alpha}B_nf(x)=\sum_{j=0}^{\alpha}\frac{(-1)^j}{n^j j!}\left.\frac{\partial^j P_n^{\langle s\rangle}f(x)}{\partial s^j}\right|_{s=0}\qquad (f\colon[0,1]\to\mathbf{R},\ x\in[0,1]);$$

that is, ${}_{\alpha}B_nf(x)$ (for fixed f, x) is generated by putting $s=-1/n$ in the Maclaurin series truncated at degree α of $P_n^{\langle s\rangle}f(x)$ regarded as a function of s.

Remark. The function $P_n^{\langle s\rangle} f(x)$ (with respect to s) is analytic where $(n-1)\,|s|<1$ because

$$|1+\mu s| \geqslant 1-\mu\,|s| \geqslant 1-(n-1)\,|s|>0 \qquad (\mu=0, 1, ..., n-1),$$

and $s=-1/n$ belongs to the region $(n-1)\,|s|<1$.

Note that the identities ${}_0B_n=P_n^{\langle 0\rangle}=B_n$ and ${}_\infty B_n=P_n^{\langle -1/n\rangle}=L_n$ hold. In this context, ${}_\alpha B_n$ can be regarded as an "intermediate" operator between B_n and L_n. Though Sablonnière's operator $B_n^{(K)}$ is also an intermediate one between them, our new operator is substantially distinct from it. (Now we present Table II for values of $\|{}_\alpha B_n\|$, which will be compared with Table I later.)

2.2. *Representations and Properties of ${}_\alpha B_n$*

Now we provide two kinds of representations of our operator.

THEOREM 2.1. *The modified Bernstein operator ${}_\alpha B_n$ has the representation*

$$\begin{aligned} {}_\alpha B_n f &= \sum_{k=0}^{2\alpha} (B_n f)^{[k]} \sum_{j=0}^{\alpha} \frac{\Upsilon_{j,k}}{n^j} \\ &= \sum_{j=0}^{\alpha} \frac{1}{n^j} \sum_{k=0}^{2j} \Upsilon_{j,k}(B_n f)^{[k]} \qquad (f\colon [0,1] \to \mathbf{R}), \end{aligned}$$

where $\Upsilon_{j,k}$ are the polynomials of degree at most k determined by the recursion formula

TABLE II

Approximate Values of $\|{}_\alpha B_n\|$

	α							
n	1	2	3	4	5	6	7	8
2	1.083	1.161	1.204	1.227	1.238	1.244	1.247	1.249
3	1.089	1.212	1.334	1.427	1.493	1.538	1.569	1.589
4	1.125	1.234	1.405	1.567	1.708	1.824	1.915	1.986
5	1.125	1.250	1.451	1.665	1.879	2.078	2.255	2.408
6	1.134	1.261	1.483	1.735	2.012	2.296	2.574	2.834
7	1.133	1.269	1.506	1.789	2.118	2.482	2.862	3.247
8	1.133	1.275	1.524	1.831	2.204	2.639	3.120	3.637
9	1.136	1.280	1.538	1.865	2.276	2.775	3.351	4.000
10	1.135	1.283	1.549	1.893	2.337	2.891	3.556	4.335
16	1.138	1.296	1.589	1.992	2.563	3.349	4.415	5.849
32	1.140	1.306	1.622	2.081	2.777	3.819	5.379	7.739

$$\Upsilon_{j,-1} = 0 \qquad (j \geqslant 0),$$

$$\Upsilon_{0,0} = 1,$$

$$\Upsilon_{j,0} = 0 \qquad (j \geqslant 1),$$

$$\Upsilon_{0,k} = 0 \qquad (1 \leqslant k \leqslant n),$$

$$\Upsilon_{j,k+1} = k(\Upsilon_{j-1,k+1} - e_1 \Upsilon_{j-1,k} - e_2 \Upsilon_{j-1,k-1})$$
$$(j \geqslant 1, 0 \leqslant k \leqslant n-1),$$

where $e_1(x) = 1 - 2x$, $e_2(x) = x(1-x)$.

Remark 1. In this paper, we generally define the polynomials e_p $(p \in \mathbf{N}_0)$ as $e_{2p}(x) = (x(1-x))^p$, $e_{2p+1}(x) = (1-2x)\, e_{2p}(x)$.

Remark 2. Since $\Upsilon_{j,k}$ are independent of n and we can take n arbitrarily large, we can understand that $\Upsilon_{j,k}$ are defined for all $j, k \in \mathbf{N}_0$.

THEOREM 2.2. *The modified Bernstein operator* ${}_{\alpha}B_n$ *has the explicit representation*

$$ {}_{\alpha}B_n f(x) = \sum_{v=0}^{n} f\left(\frac{v}{n}\right) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{kx}{v} \binom{k(1-x)}{n-v} \left(\frac{k}{n}\right)^{\alpha},$$
$$(f \colon [0,1] \to \mathbf{R},\ x \in [0,1]).$$

Note that we can extend the definition of ${}_{\alpha}B_n$ for all nonnegative real numbers α by using this theorem. This is a surprising fact.

The following theorem, which concerns stability and convergence rate, is the theoretically most important result in this paper.

THEOREM 2.3. *For each* $\alpha \in \mathbf{N}_0$, *the sequence* $\{{}_{\alpha}B_n\}_{n=1}^{\infty}$ *has the following properties*:

(1) *for all* $p, q, r \in \mathbf{N}_0$, *there exists a constant* M *such that for all* $n \in \mathbf{N}$ *and for all* $f \in C^r[0,1]$,

$$\|e_{2p}({}_{\alpha}B_n f)^{(q+r)}\| \leqslant M n^{q - \min\{p, [q/2]\}} \|f^{(r)}\|;$$

(2) *for all* $\beta, \gamma \in \mathbf{N}_0$ $(\beta \leqslant \alpha)$ *and for all* $f \in C^{2\beta+\gamma}[0,1]$,

$$\|({}_{\alpha}B_n f)^{(\gamma)} - f^{(\gamma)}\| = o(n^{-\beta}) \qquad (n \to \infty);$$

(3) *for all* $\gamma \in \mathbf{N}_0$ *and for all* $f \in C^{2\alpha+\gamma+2}[0, 1]$,

$$\lim_{n \to \infty} n^{\alpha+1}(({}_\alpha B_n f)^{(\gamma)} - f^{(\gamma)}) = -\left(\sum_{k=0}^{2\alpha+2} \Upsilon_{\alpha+1, k} f^{[k]}\right)^{(\gamma)} \qquad \textit{in the sense of } \|\cdot\|;$$

where $\|\cdot\|$ *is the uniform norm on* $C[0, 1]$.

2.3. *Application to Numerical Quadrature*

We denote by I the integration operator defined as

$$If = \int_0^1 f(x)\, dx \qquad (f \in C[0, 1]),$$

and we define the integrating operator ${}_\alpha I_n$ as

$${}_\alpha I_n = I_\alpha B_n.$$

Theorem 2.3 readily implies the following corollary.

COROLLARY 2.1. *For each* $\alpha \in \mathbf{N}_0$, *the sequence* $\{{}_\alpha I_n\}_{n=1}^{\infty}$ *has the following properties*:

(1) *there exists a constant* M *such that for all* $n \in \mathbf{N}$ *and for all* $f \in C[0, 1]$,

$$|{}_\alpha I_n f| \leqslant M \|f\|;$$

(2) *for all* $\beta \in \mathbf{N}_0$ $(\beta \leqslant \alpha)$ *and for all* $f \in C^{2\beta}[0, 1]$,

$$|{}_\alpha I_n f - If| = o(n^{-\beta}) \qquad (n \to \infty);$$

(3) *for all* $f \in C^{2\alpha+2}[0, 1]$,

$$\lim_{n \to \infty} n^{\alpha+1}({}_\alpha I_n f - If) = -I \sum_{k=0}^{2\alpha+2} \Upsilon_{\alpha+1, k} f^{[k]}.$$

Now let us consider how to calculate ${}_\alpha I_n f$. It is a detour to represent ${}_\alpha B_n f$ as the form in Theorem 2.1 and then integrate it on $[0, 1]$. In fact, there is a more direct way to calculate ${}_\alpha I_n f$.

THEOREM 2.4. *The integrating operator* ${}_\alpha I_n$ *has the representation*

$${}_\alpha I_n f = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) {}_\alpha w_{n, \nu},$$

where

$$
{}_0w_{n,\nu} = \frac{1}{n+1},
$$

$$
{}_\alpha w_{n,\nu} = \frac{1}{n}\Bigg[1 - \sum_{k=0}^{\alpha-1} (-1)^k \gamma_{k+1} \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu}\right) \times n^{(k)} \sum_{j=k}^{\alpha-1} \frac{S(j-1, k-1)}{n^j}\Bigg] \qquad (\alpha \geqslant 1),
$$

where $\gamma_k = \int_0^1 \binom{x}{k}\, dx$ $(k \in \mathbf{N}_0)$ *and* $S(j,k)$ *are the Stirling numbers of the second kind with the conventional definition* $S(-1,-1) = 1$, $S(j,-1) = 0$ $(j \geqslant 0)$.

Remark 1. The symbol $n^{(k)}$ stands for $\prod_{\mu=0}^{k-1} (n-\mu)$.

Remark 2. There are various notations for the Stirling numbers of the second kind [1, p. 822], but we adopt the symbol $S(\cdot,\cdot)$ because it has often been used recently (e.g., [10, Chapter 13]).

Let us denote by $\mathscr{B}_\nu^{(n)}(\cdot)$ the Bernoulli polynomial of order n and degree ν [4, pp. 124–127]. (We use the symbol $\mathscr{B}$ instead of B to distinguish the Bernoulli polynomials from the Bernstein operators.) Then the identities $\gamma_k = \mathscr{B}_k^{(k)}(1)/k!$ and $S(j,k) = \binom{j}{k} \mathscr{B}_{j-k}^{(-k)}(0)$ hold [4, pp. 130, 133]. It is interesting that these two systems of numbers are unified in terms of the Bernoulli polynomials.

Though we can calculate γ_k's by way of the Bernoulli polynomials, there is a more direct way to calculate them. Let $0 < |t| < 1$ and consider their generating function

$$
\sum_{k=0}^{\infty} \gamma_k t^k = \int_0^1 (1+t)^x\, dx = \frac{t}{\log(1+t)}.
$$

(An equivalent identity for $\mathscr{B}_k^{(k)}(1)$ appears in [4, p. 135].) Since we can expand

$$
\frac{\log(1+t)}{t} = \sum_{l=0}^{\infty} \frac{(-t)^l}{l+1},
$$

we can calculate

$$
\left(\sum_{l=0}^{\infty} \frac{(-t)^l}{l+1}\right)\left(\sum_{k=0}^{\infty} \gamma_k t^k\right) = \sum_{k=0}^{\infty} t^k \sum_{l=0}^{k} \frac{(-1)^l \gamma_{k-l}}{l+1} = 1.
$$

Equating coefficients of t^k, we obtain the recursion formula

$$\gamma_0 = 1, \qquad \gamma_k = \sum_{l=1}^{k} \frac{(-1)^{l-1} \gamma_{k-l}}{l+1} \qquad (k \geqslant 1).$$

Some γ_k's are evaluated as follows:

$$\gamma_1 = \frac{1}{2}, \qquad \gamma_2 = -\frac{1}{12}, \qquad \gamma_3 = \frac{1}{24}, \qquad \gamma_4 = -\frac{19}{720}, \qquad \gamma_5 = \frac{3}{160}.$$

Now let Δ_h, ∇_h be the forward and backward difference operators, respectively, of stepsize h ($h \in \mathbf{R}$, $h > 0$). Then the above theorem and the identity

$$\begin{aligned}
&(-1)^k \Delta^k_{1/n} f(0) + \nabla^k_{1/n} f(1) \\
&\quad = (-1)^k \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right) + \sum_{\nu=0}^{k} (-1)^{\nu} \binom{k}{\nu} f\left(1 - \frac{\nu}{n}\right) \\
&\quad = \sum_{\nu=0}^{k} (-1)^{\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right) + \sum_{\nu=n-k}^{n} (-1)^{n-\nu} \binom{k}{n-\nu} f\left(\frac{\nu}{n}\right) \\
&\quad = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \left((-1)^{\nu} \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu}\right)
\end{aligned}$$

imply the following corollary.

COROLLARY 2.2. *When* $\alpha \geqslant 1$, *the integrating operator* ${}_\alpha I_n$ *can be represented as*

$$\begin{aligned}
{}_\alpha I_n f = \frac{1}{n} \Bigg[&\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) - \sum_{k=0}^{\alpha-1} \gamma_{k+1} (\Delta^k_{1/n} f(0) + (-1)^k \nabla^k_{1/n} f(1))\, n^{(k)} \\
&\times \sum_{j=k}^{\alpha-1} \frac{S(j-1, k-1)}{n^j} \Bigg].
\end{aligned}$$

This corollary means that ${}_\alpha I_n (\alpha \geqslant 1)$ brings us a kind of trapezoidal rule with end modifications. Particularly when $\alpha = 1$, it coincides with the ordinary trapezoidal rule.

Now let us consider positivity of the linear operator ${}_\alpha I_n$. Theorem 2.1 (or, Proposition 3.1 and Definition 2.1) readily implies ${}_\alpha B_n f = f$ for all linear functions f, and in particular, ${}_\alpha I_n 1 = 1$. Therefore, as long as the linear operator ${}_\alpha I_n$ is positive, the identity $\| {}_\alpha I_n \| = 1$ holds, which means ${}_\alpha I_n$ is the best for stability. On the other hand, from the viewpoint of convergence rate, it is desirable to choose α as large as possible. For this reason, we naturally become interested in the problem of determining the maximum of α that preserves positivity of ${}_\alpha I_n$ for each n. We have the following result.

THEOREM 2.5. *For each $n \in \mathbf{N}$, we define $\alpha(n) = \max\{\alpha \in \mathbf{N}_0 \cup \{\infty\} \mid {}_\alpha I_n$ is positive$\}$. Then*

$$\alpha(n) = \begin{cases} \infty & (1 \leqslant n \leqslant 7,\ n = 9) \\ 13 & (n = 8,\ 10,\ 11) \\ 12 & (n = 12) \\ 11 & (13 \leqslant n \leqslant 15) \\ 10 & (16 \leqslant n \leqslant 23) \\ 9 & (24 \leqslant n \leqslant 104) \\ 8 & (n \geqslant 105). \end{cases}$$

Furthermore, all the ${}_\alpha I_n$ ($\alpha < \alpha(n)$, $\alpha \in \mathbf{N}_0$) are positive.

Remark. The operator ${}_\infty I_n$ means that

$${}_\infty I_n f = \int_0^1 {}_\infty B_n f(x)\, dx = \int_0^1 L_n f(x)\, dx;$$

that is to say, ${}_\infty I_n$ is the operator corresponding to the Newton–Cotes rule.

We present Table III, which is a result of numerical experiments on the Runge function $f(x) = 1/(1 + 25(2x-1)^2)$. This suggests that our "intermediate" rule between the trapezoidal and the Newton–Cotes ones is very effective for numerical quadrature.

TABLE III

Approximate Values of $|{}_\alpha I_n f - If|$ When $f(x) = 1/(1+25(2x-1)^2)$

	α			
n	1	8	9	∞
4	5.39×10^{-2}	1.30×10^{-2}	1.89×10^{-2}	3.73×10^{-2}
8	3.77×10^{-3}	4.61×10^{-3}	1.08×10^{-2}	1.25×10^{-1}
16	6.90×10^{-5}	1.83×10^{-5}	4.11×10^{-5}	8.99×10^{-1}
32	2.41×10^{-5}	1.48×10^{-9}	1.22×10^{-9}	1.51×10^{2}
64	6.02×10^{-6}	3.34×10^{-13}	3.75×10^{-14}	1.51×10^{7}
128	1.50×10^{-6}	4.48×10^{-16}	2.73×10^{-17}	—
256	3.76×10^{-7}	7.24×10^{-19}	2.21×10^{-20}	—

3. PROPOSITIONS REGARDING STANCU'S OPERATOR

Before proving the main results, here we collect several propositions about Stancu's operator, most of which are new.

PROPOSITION 3.1. *Stancu's operator $P_n^{\langle s \rangle}$ has the following representation in terms of the difference operators*:

$$P_n^{\langle s \rangle} f(x) = \sum_{r=0}^{n} \Delta_{1/n}^r f(0) \binom{n}{r} \prod_{\mu=0}^{r-1} \frac{x+\mu s}{1+\mu s} \tag{3.1}$$

$$= \sum_{r=0}^{n} (-1)^r \nabla_{1/n}^r f(1) \binom{n}{r} \prod_{\mu=0}^{r-1} \frac{1-x+\mu s}{1+\mu s}. \tag{3.2}$$

Remark. In particular, by considering the cases $s=0$ and $s=-1/n$, we obtain

$$B_n f(x) = \sum_{r=0}^{n} \Delta_{1/n}^r f(0) \binom{n}{r} x^r = \sum_{r=0}^{n} (-1)^r \nabla_{1/n}^r f(1) \binom{n}{r} (1-x)^r,$$

$$L_n f(x) = \sum_{r=0}^{n} \Delta_{1/n}^r f(0) \binom{nx}{r} = \sum_{r=0}^{n} (-1)^r \nabla_{1/n}^r f(1) \binom{n(1-x)}{r}.$$

The latter is nothing but Newton's forward (backward) interpolation formula.

Proofs of this proposition appear in [8, 9], but here we give a more direct one.

Proof of Proposition 3.1. Suppose $s \neq 0$ and let $a = -1/s$. Then the right-hand side of (3.1) can be calculated as

$$\begin{aligned}
\sum_{r=0}^{n} \Delta_{1/n}^r f(0) \binom{n}{r} \frac{(ax)^{(r)}}{a^{(r)}} &= \sum_{r=0}^{n} \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f\left(\frac{\nu}{n}\right) \binom{n}{r} \frac{(ax)^{(r)}}{a^{(r)}} \\
&= \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \sum_{r=\nu}^{n} (-1)^{r-\nu} \binom{n-\nu}{r-\nu} \frac{(ax)^{(r)}}{a^{(r)}} \\
&= \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \sum_{r=0}^{n-\nu} (-1)^r \binom{n-\nu}{r} \frac{(ax)^{(r+\nu)}}{a^{(r+\nu)}} \\
&= \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \binom{n}{\nu} \frac{(ax)^{(\nu)}}{a^{(n)}} \sum_{r=0}^{n-\nu} (-1)^r \binom{n-\nu}{r} \\
&\quad \times (ax-\nu)^{(r)} (a-r-\nu)^{(n-r-\nu)}.
\end{aligned}$$

By using the identities $a^{(m)}=(-1)^m(-a+m-1)^{(m)}$ and $\sum_{r=0}^{m}\binom{m}{r} a^{(r)}b^{(m-r)}=(a+b)^{(m)}$, the inner sum can be calculated as

$$\begin{aligned}
&\sum_{r=0}^{n-v}(-1)^r\binom{n-v}{r}(ax-v)^{(r)}(-1)^{n-r-v}(-a+n-1)^{(n-r-v)}\\
&\quad=(-1)^{n-v}\sum_{r=0}^{n-v}\binom{n-v}{r}(ax-v)^{(r)}(-a+n-1)^{(n-r-v)}\\
&\quad=(-1)^{n-v}(-a(1-x)+n-v-1)^{(n-v)}=(a(1-x))^{(n-v)}.
\end{aligned}$$

Thus the right-hand side of (3.1) equals $P_n^{\langle s\rangle}f(x)$ if $s\neq 0$. Since both sides of (3.1) are continuous at $s=0$, (3.1) is valid also when $s=0$. Letting $\tilde{f}(x)=f(1-x)$ and replacing f by $\tilde{f}$ and x by $1-x$ imply (3.2). ∎

PROPOSITION 3.2. *Stancu's operator $P_n^{\langle s\rangle}$ has the degree-preserving property*

$$P_n^{\langle s\rangle}\mathbf{P}_m\subseteq\mathbf{P}_m \qquad (0\leqslant m\leqslant n),$$

where $\mathbf{P}_m$ is the set of polynomials of degree at most $m\in\mathbf{N}_0$ with real coefficients.

Proof. Let $f\in\mathbf{P}_m$. Then $\Delta_{1/n}^r f(0)=0$ if $r>m$. Proposition 3.1 implies $P_n^{\langle s\rangle}f\in\mathbf{P}_m$. ∎

PROPOSITION 3.3. *There exist unique $U_k^{\langle s\rangle}\in\mathbf{P}_k(0\leqslant k\leqslant n)$ such that*

$$P_n^{\langle s\rangle}f=\sum_{k=0}^{n}U_k^{\langle s\rangle}(B_nf)^{[k]} \qquad (f\colon[0,1]\to\mathbf{R}),$$

where $U_k^{\langle s\rangle}$ are determined by the recursion formula

$$\begin{gathered}
U_{-1}^{\langle s\rangle}=0, \qquad U_0^{\langle s\rangle}=1,\\
(1+ks)\,U_{k+1}^{\langle s\rangle}=ks(e_1U_k^{\langle s\rangle}+e_2U_{k-1}^{\langle s\rangle}) \quad (0\leqslant k\leqslant n-1).
\end{gathered}$$

Note that this proposition is a generalization of Theorem 2.2 in [3], which lies in the special case $s=-1/n$. (We can identify $U_k^{\langle -1/n\rangle}$ with U_k^n in (1.1).)

Proof of Proposition 3.3. The unique existence of $U_k^{\langle s\rangle}\in\mathbf{P}_k$ is guaranteed by Theorem 2.1 in [3] and Proposition 3.2. It suffices to derive the recursion formula. Since it is obviously valid when $s=0$, we suppose $s\neq 0$ and let $a=-1/s$.

Let $x \in [0, 1]$, $t \in (-1, 1)$ and fix them for a while. We consider the case

$$f(\xi) = (1 + (1-x)\, t)^{n\xi}\, (1 - xt)^{n(1-\xi)} \qquad (\xi \in [0, 1]).$$

Then, as we did in the proof of Theorem 2.1 in [3], we get

$$(B_n f)^{[k]}(x) = \binom{n}{k} t^k \qquad \text{for all} \quad k \leqslant n.$$

Therefore the relation $P_n^{\langle s \rangle} f = \sum_{k=0}^{n} U_k^{\langle s \rangle} (B_n f)^{[k]}$ implies

$$\sum_{v=0}^{n} (1 + (1-x)\, t)^{v}\, (1 - xt)^{n-v} \binom{n}{v} \frac{(ax)^{(v)}\, (a(1-x))^{(n-v)}}{a^{(n)}}$$
$$= \sum_{k=0}^{n} \binom{n}{k} U_k^{\langle s \rangle}(x)\, t^k. \tag{3.3}$$

The left-hand side can be expanded as

$$\sum_{v=0}^{n} \left[\sum_{l=0}^{v} \binom{v}{l} (1-x)^l\, t^l \right] \left[\sum_{m=0}^{n-v} \binom{n-v}{m} (-x)^m\, t^m \right] \binom{ax}{v} \binom{a(1-x)}{n-v} \binom{a}{n}^{-1}.$$

Since the region $0 \leqslant v \leqslant n$, $0 \leqslant l \leqslant v$, $0 \leqslant m \leqslant n-v$ corresponds to the region $0 \leqslant k \leqslant n$, $0 \leqslant l \leqslant k$, $0 \leqslant v' \leqslant n-k$ when we let $k = l + m$, $v' = v - l$, the above formula equals

$$\sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{v=0}^{n-k} t^k (1-x)^l\, (-x)^{k-l} \binom{v+l}{l} \binom{ax}{v+l} \binom{n-v-l}{k-l}$$
$$\times \binom{a(1-x)}{n-v-l} \binom{a}{n}^{-1}$$
$$= \sum_{k=0}^{n} t^k \binom{a}{n}^{-1} \sum_{l=0}^{k} \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l\, (-x)^{k-l}$$
$$\times \sum_{v=0}^{n-k} \binom{ax-l}{v} \binom{a(1-x)-k+l}{n-k-v}$$
$$= \sum_{k=0}^{n} t^k \binom{a}{n}^{-1} \sum_{l=0}^{k} \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l\, (-x)^{k-l} \binom{a-k}{n-k}$$
$$= \sum_{k=0}^{n} t^k \binom{n}{k} \binom{a}{k}^{-1} \sum_{l=0}^{k} \binom{ax}{l} \binom{a(1-x)}{k-l} (1-x)^l\, (-x)^{k-l}.$$

Therefore, equating coefficients of t^k on both sides of (3.3) yields

$$\binom{a}{k} U_k^{\langle s\rangle}(x) = \sum_{l=0}^{k} \binom{ax}{l}\binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l} \qquad (0 \leqslant k \leqslant n).$$

Here we let $\tilde{U}_k(x)$ be the right-hand side for all $k \in \mathbf{N}_0$ and consider its generating function

$$\begin{aligned}\sum_{k=0}^{\infty} \tilde{U}_k(x)\, t^k &= \sum_{k=0}^{\infty} t^k \sum_{l=0}^{k} \binom{ax}{l}\binom{a(1-x)}{k-l} (1-x)^l (-x)^{k-l} \\ &= \left[\sum_{l=0}^{\infty} \binom{ax}{l} (1-x)^l\, t^l\right]\left[\sum_{m=0}^{\infty} \binom{a(1-x)}{m} (-x)^m\, t^m\right] \\ &= (1+(1-x)\, t)^{ax}\, (1-xt)^{a(1-x)}. \qquad (3.4)\end{aligned}$$

Putting $t=0$ gives

$$\tilde{U}_0 = U_0^{\langle s\rangle} = 1. \qquad (3.5)$$

Differentiating (3.4) by t and multiplying it by $(1+(1-x)\,t)(1-xt)$, we get

$$(1+e_1(x)\, t - e_2(x)\, t^2) \sum_{k=1}^{\infty} k\tilde{U}_k(x)\, t^{k-1} = -ae_2(x)\, t \sum_{k=0}^{\infty} \tilde{U}_k(x)\, t^k$$

by virtue of

$$\begin{aligned}&(1+(1-x)\, t)(1-xt) \frac{d}{dt} [(1+(1-x)\, t)^{ax}(1-xt)^{a(1-x)}] \\ &\quad = -ae_2(x)\, t(1+(1-x)\, t)^{ax}\, (1-xt)^{a(1-x)}.\end{aligned}$$

Rearrangement of the above formula gives

$$\begin{aligned}&\sum_{k=0}^{\infty} (k+1)\, \tilde{U}_{k+1}(x)\, t^k \\ &\quad = -e_1(x) \sum_{k=0}^{\infty} k\tilde{U}_k(x)\, t^k - e_2(x) \sum_{k=1}^{\infty} (a-k+1)\, \tilde{U}_{k-1}(x)\, t^k.\end{aligned}$$

Equating coefficients of t^k $(0 \leqslant k \leqslant n-1)$ on both sides yields

$$\tilde{U}_1 = 0, \quad (k+1)\, \tilde{U}_{k+1} = -ke_1 \tilde{U}_k - (a-k+1)\, e_2 \tilde{U}_{k-1} \quad (1 \leqslant k \leqslant n-1).$$

Recalling that $\widetilde{U}_k = \binom{a}{k} U_k^{\langle s \rangle}$ $(0 \leqslant k \leqslant n)$ and remarking that the stipulation $\prod_{\mu=0}^{n-1}(1+\mu s) \neq 0$ implies $\binom{a}{k} \neq 0$, we obtain

$$U_1^{\langle s \rangle} = 0, \qquad (a-k)\, U_{k+1}^{\langle s \rangle} = -k(e_1 U_k^{\langle s \rangle} + e_2 U_{k-1}^{\langle s \rangle}) \quad (1 \leqslant k \leqslant n-1);$$

that is,

$$U_{-1}^{\langle s \rangle} = 0, \qquad U_0^{\langle s \rangle} = 1,$$

$$(1+ks)\, U_{k+1}^{\langle s \rangle} = ks(e_1 U_k^{\langle s \rangle} + e_2 U_{k-1}^{\langle s \rangle}) \qquad (0 \leqslant k \leqslant n-1),$$

where $U_{-1}^{\langle s \rangle} = 0$ is a conventional definition and we used (3.5). ∎

Proposition 3.4. *Stancu's operator* $P_n^{\langle s \rangle}$ *can be represented as*

$$P_n^{\langle s \rangle} f(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\left[\binom{nx}{\nu}\binom{n(1-x)}{n-\nu} + (1+ns)\sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{1+ks}\binom{n}{k}\binom{kx}{\nu}\binom{k(1-x)}{n-\nu}\right]. \tag{3.6}$$

Remark. This proposition signifies that we can take out the factor $1+ns$ from $P_n^{\langle s \rangle} - L_n$, as is expected from the identity $P_n^{\langle -1/n \rangle} = L_n$.

Proof. Assume $s \neq 0$ and let $a = -1/s$. Then

$$P_n^{\langle s \rangle} f(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{ax}{\nu}\binom{a(1-x)}{n-\nu}\binom{a}{n}^{-1}. \tag{3.7}$$

Since

$$\lim_{a \to \infty} \binom{ax}{\nu}\binom{a(1-x)}{n-\nu}\binom{a}{n}^{-1} = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$$

and

$$\lim_{a \to k} (a-k)\binom{ax}{\nu}\binom{a(1-x)}{n-\nu}\binom{a}{n}^{-1}$$

$$= \frac{n!}{\left[\prod_{\mu=0}^{k-1}(k-\mu)\right]\left[\prod_{\mu=k+1}^{n-1}(k-\mu)\right]}\binom{kx}{\nu}\binom{k(1-x)}{n-\nu}$$

$$= -(-1)^{n-k}(n-k)\binom{n}{k}\binom{kx}{\nu}\binom{k(1-x)}{n-\nu},$$

we can decompose into partial fractions with respect to a,

$$\binom{ax}{v}\binom{a(1-x)}{n-v}\binom{a}{n}^{-1}$$
$$=\binom{n}{v}x^v(1-x)^{n-v}-\sum_{k=0}^{n-1}\frac{(-1)^{n-k}(n-k)}{a-k}\binom{n}{k}\binom{kx}{v}\binom{k(1-x)}{n-v}.$$

Putting $a=n$ on both sides yields

$$\binom{nx}{v}\binom{n(1-x)}{n-v}=\binom{n}{v}x^v(1-x)^{n-v}-\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\binom{kx}{v}\binom{k(1-x)}{n-v}. \tag{3.8}$$

Eliminating $\binom{n}{v}x^v(1-x)^{n-v}$ from the above two formulas gives

$$\binom{ax}{v}\binom{a(1-x)}{n-v}\binom{a}{n}^{-1}$$
$$=\binom{nx}{v}\binom{n(1-x)}{n-v}+\sum_{k=0}^{n-1}\frac{(-1)^{n-k}(a-n)}{a-k}\binom{n}{k}\binom{kx}{v}\binom{k(1-x)}{n-v}.$$

Equation (3.7), this identity, and $a=-1/s$ imply (3.6) when $s\neq 0$. It is valid also when $s=0$ because both sides of (3.6) are continuous at $s=0$. ∎

Let $\varphi(k)=\binom{kx}{v}\binom{k(1-x)}{n-v}$. Then $\varphi\in\mathbf{P}_n$ and its leading coefficient is $x^v(1-x)^{n-v}/(v!(n-v)!)$. Hence we have

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{kx}{v}\binom{k(1-x)}{n-v}=\Delta_1^n\varphi(0)=\binom{n}{v}x^v(1-x)^{n-v}.$$

Note that this identity gives another proof of (3.8) and indicates that Theorem 2.2 is valid when $\alpha=0$.

PROPOSITION 3.5. *Stancu's operator $P_n^{\langle s\rangle}$ satisfies the identity*

$$\int_0^1 P_n^{\langle s\rangle}f(x)\,dx=\sum_{v=0}^{n}f\left(\frac{v}{n}\right)\Bigg[\frac{1-s}{n+1}+\sum_{k=0}^{n}\gamma_{k+1}\left((-1)^v\binom{k}{v}+(-1)^{n-v}\binom{k}{n-v}\right)\frac{n^{(k)}s^{k+1}}{\prod_{\mu=0}^{k-1}(1+\mu s)}\Bigg]. \tag{3.9}$$

Remark. When $s=-1/n$, the above identity reduces to

$$\int_0^1 L_n f(x)\,dx = \frac{1}{n}\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\left[1-\sum_{k=0}^{n}(-1)^k \gamma_{k+1} \times\left((-1)^\nu\binom{k}{\nu}+(-1)^{n-\nu}\binom{k}{n-\nu}\right)\right]. \tag{3.10}$$

This is nothing but the Newton–Cotes rule, all the weights in which are represented explicitly.

Proof. We first prove that for all $a\in\mathbf{R}$ and for all $\nu,\mu\in\mathbf{N}_0$,

$$\int_0^a \binom{x}{\nu}\binom{a-x}{\mu}dx = \binom{a+1}{\nu+\mu+1}-\sum_{k=0}^{\nu+\mu}(-1)^k\gamma_{k+1} \times\binom{a-k}{\nu+\mu-k}\left((-1)^\nu\binom{k}{\nu}+(-1)^\mu\binom{k}{\mu}\right) \tag{3.11}$$

by using the double generating functions of both sides. Let $|u|,|v|<1/3$, and $u\neq v$. Then we can calculate

$$\begin{aligned}
\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}u^\nu v^\mu \int_0^a \binom{x}{\nu}\binom{a-x}{\mu}dx
&=\int_0^a (1+u)^x(1+v)^{a-x}\,dx\\
&=(1+v)^a\int_0^a\left(\frac{1+u}{1+v}\right)^x dx\\
&=\frac{(1+u)^a-(1+v)^a}{\log(1+u)-\log(1+v)};
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}u^\nu v^\mu\binom{a+1}{\nu+\mu+1}
&=\sum_{n=1}^{\infty}\sum_{\nu=0}^{n-1}u^\nu v^{n-\nu-1}\binom{a+1}{n}\\
&=\sum_{n=1}^{\infty}\binom{a+1}{n}\frac{u^n-v^n}{u-v}\\
&=\frac{(1+u)^{a+1}-(1+v)^{a+1}}{u-v};
\end{aligned}\tag{3.13}$$

$$\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} u^\nu v^\mu \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} (-1)^\nu \binom{k}{\nu}$$

$$= \sum_{k=0}^{\infty}\sum_{\nu=0}^{k}\sum_{n=0}^{\infty} u^\nu v^{n+k-\nu} (-1)^k \gamma_{k+1} \binom{a-k}{n} (-1)^\nu \binom{k}{\nu}$$

$$= \sum_{k=0}^{\infty} \gamma_{k+1} \left[\sum_{\nu=0}^{k} \binom{k}{\nu} u^\nu (-v)^{k-\nu}\right]\left[\sum_{n=0}^{\infty} \binom{a-k}{n} v^n\right]$$

$$= \sum_{k=0}^{\infty} \gamma_{k+1} (u-v)^k \, (1+v)^{a-k}$$

$$= \sum_{k=1}^{\infty} \gamma_k (u-v)^{k-1} \, (1+v)^{a-k+1}$$

$$= \frac{(1+v)^{a+1}}{u-v} \int_0^1 \left[\sum_{k=1}^{\infty} \binom{x}{k} \left(\frac{u-v}{1+v}\right)^k\right] dx$$

$$= \frac{(1+v)^{a+1}}{u-v} \int_0^1 \left[\left(1+\frac{u-v}{1+v}\right)^x - 1\right] dx$$

$$= \frac{(1+v)^a}{\log(1+u)-\log(1+v)} - \frac{(1+v)^{a+1}}{u-v}, \tag{3.14}$$

where we noticed $|(u-v)/(1+v)| \leqslant (|u|+|v|)/(1-|v|) < 1$. Applying this identity, we have also

$$\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} u^\nu v^\mu \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} (-1)^\mu \binom{k}{\mu}$$

$$= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} v^\nu u^\mu \sum_{k=0}^{\nu+\mu} (-1)^k \gamma_{k+1} \binom{a-k}{\nu+\mu-k} (-1)^\nu \binom{k}{\nu}$$

$$= \frac{(1+u)^a}{\log(1+v)-\log(1+u)} - \frac{(1+u)^{a+1}}{v-u}. \tag{3.15}$$

Since $(3.13)-((3.14)+(3.15))$ is equal to (3.12), equating coefficients of $u^\nu v^\mu$ yields (3.11). Putting $\mu = n-\nu$, we obtain

$$\int_0^a \binom{x}{\nu}\binom{a-x}{n-\nu} dx = \binom{a+1}{n+1} - \sum_{k=0}^{n} (-1)^k \gamma_{k+1}$$
$$\times \binom{a-k}{n-k}\left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu} \binom{k}{n-\nu}\right).$$

If $a^{(n)} \neq 0$ then we get

$$\int_0^1 \binom{ax}{\nu}\binom{a(1-x)}{n-\nu} a\,dx = \frac{a+1}{n+1}\binom{a}{n} - \sum_{k=0}^{n} (-1)^k \gamma_{k+1} \times \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu}\binom{k}{n-\nu}\right) \frac{n^{(k)}}{a^{(k)}}\binom{a}{n};$$

that is,

$$\int_0^1 \binom{ax}{\nu}\binom{a(1-x)}{n-\nu}\binom{a}{n}^{-1} dx = \frac{a+1}{(n+1)\,a} - \sum_{k=0}^{n} (-1)^k \gamma_{k+1} \times \left((-1)^\nu \binom{k}{\nu} + (-1)^{n-\nu}\binom{k}{n-\nu}\right) \frac{n^{(k)}}{a \cdot a^{(k)}}.$$

Therefore, putting $a = -1/s$ and (3.7) imply (3.9) when $s \neq 0$. It is valid also when $s = 0$ because both sides of it are continuous at $s = 0$. ∎

4. PROOFS OF THE MAIN RESULTS

Now we are to prove all the theorems given in Section 2.

Proof of Theorem 2.1. When $(n-1)\,|s| < 1$, we can expand for each $x \in [0, 1]$

$$U_k^{\langle s \rangle}(x) = \sum_{j=0}^{\infty} \Upsilon_{j,k}(x)(-s)^j \qquad (-1 \leqslant k \leqslant n). \tag{4.1}$$

The recursion formula in Proposition 3.3 immediately implies

$$\Upsilon_{j,-1} = 0 \quad (j \geqslant 0), \qquad \Upsilon_{0,0} = 1, \qquad \Upsilon_{j,0} = 0 \quad (j \geqslant 1).$$

Furthermore, for every k satisfying $0 \leqslant k \leqslant n-1$, it also implies

$$(1+ks) \sum_{j=0}^{\infty} \Upsilon_{j,k+1}(x)(-s)^j = ks\left(e_1(x) \sum_{j=0}^{\infty} \Upsilon_{j,k}(x)(-s)^j + e_2(x) \sum_{j=0}^{\infty} \Upsilon_{j,k-1}(x)(-s)^j\right).$$

Rearrangement of this formula gives

$$\sum_{j=0}^{\infty} \Upsilon_{j,k+1}(x)(-s)^j$$
$$= k \sum_{j=1}^{\infty} (\Upsilon_{j-1,k+1}(x) - e_1(x)\, \Upsilon_{j-1,k}(x) - e_2(x)\, \Upsilon_{j-1,k-1}(x))(-s)^j.$$

Equating coefficients of $(-s)^j$ on both sides yields

$$\Upsilon_{0,k} = 0 \qquad (1 \leqslant k \leqslant n),$$
$$\Upsilon_{j,k+1} = k(\Upsilon_{j-1,k+1} - e_1 \Upsilon_{j-1,k} - e_2 \Upsilon_{j-1,k-1}) \quad (j \geqslant 1, 0 \leqslant k \leqslant n-1).$$

Thus the recursion formula in this theorem is proved. Obviously, it gives

$$\Upsilon_{j,k} = 0 \qquad (2j < k \leqslant n). \tag{4.2}$$

Therefore we obtain from Proposition 3.3 and Definition 2.1 that

$$_{\alpha}B_n f = \sum_{k=0}^{n} (B_n f)^{[k]} \sum_{j=0}^{\alpha} \frac{\Upsilon_{j,k}}{n^j} = \sum_{k=0}^{2\alpha} (B_n f)^{[k]} \sum_{j=0}^{\alpha} \frac{\Upsilon_{j,k}}{n^j}$$
$$= \sum_{j=0}^{\alpha} \frac{1}{n^j} \sum_{k=0}^{2j} \Upsilon_{j,k}(B_n f)^{[k]}.$$

In addition, it is trivial that $\Upsilon_{j,k} \in \mathbf{P}_k$. ∎

Proof of Theorem 2.2. We can calculate

$$\frac{1+ns}{1+ks} = 1 + \frac{(n-k)\,s}{1+ks} = 1 + (n-k)\,s \sum_{j=0}^{\infty} (-ks)^j.$$

If we truncate this series at degree α with respect to s, it becomes

$$1 + (n-k)\,s \sum_{j=0}^{\alpha-1} (-ks)^j = 1 + \frac{(n-k)\,s}{1+ks}(1 - (-ks)^{\alpha}).$$

If we put $s = -1/n$, this formula reduces to $(k/n)^{\alpha}$. Therefore Proposition 3.4 and Definition 2.1 imply

$$
\begin{aligned}
{}_{\alpha}B_n f(x) = \sum_{v=0}^{n} f\left(\frac{v}{n}\right)\Bigg[&\binom{nx}{v}\binom{n(1-x)}{n-v} \\
&+\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\binom{kx}{v}\binom{k(1-x)}{n-v}\left(\frac{k}{n}\right)^{\alpha}\Bigg] \\
= \sum_{v=0}^{n} f\left(\frac{v}{n}\right)&\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{kx}{v}\binom{k(1-x)}{n-v}\left(\frac{k}{n}\right)^{\alpha}. \quad \blacksquare
\end{aligned}
$$

Proof of Theorem 2.3. We deduce this theorem from Theorem 2.4 in [3]. It suffices to verify that $T_n = {}_{\alpha}B_n$ satisfies all the conditions required in the theorem.

We identify the given α with α in the theorem and let $K = 2\alpha$. Then Theorem 2.1 shows that we can let

$$V_k^n = \sum_{j=0}^{\alpha} \frac{\Upsilon_{j,k}}{n^j} \qquad (n \in \mathbf{N},\ 0 \leqslant k \leqslant n),$$

which clearly satisfies the condition (a) in Theorem 2.4 in [3]. Since (4.1) implies

$$U_k^n(x) = U_k^{\langle -1/n \rangle}(x) = \sum_{j=0}^{\infty} \frac{\Upsilon_{j,k}(x)}{n^j},$$

we can regard $V_k^n(x)$ as the asymptotic series truncated at $j = \alpha$ of $U_k^n(x)$ with respect to n. This relationship is parallel for $v_{k,l}(n)$ and $u_{k,l}(n)$ [3]. Therefore, Theorem 2.3 in [3] implies the conditions (b) and (c). In addition, it is trivial that

$$\lim_{n\to\infty} n^{\alpha+1}(V_k^n - U_k^n) = -\Upsilon_{\alpha+1,k}$$

in the sense of $\|\cdot\|$ $(0 \leqslant k \leqslant 2\alpha + 2)$. $\blacksquare$

Proof of Theorem 2.4. It is presented in [1, p. 824] and its proof is given in [2, Section 60] that the Stirling numbers of the second kind have the generating function

$$\frac{t^k}{\prod_{\mu=0}^{k}(1-\mu t)} = \sum_{j=k}^{\infty} S(j,k)\, t^j \qquad (k \in \mathbf{N}_0,\ t \in \mathbf{R},\ k\,|t| < 1).$$

This identity and the conventional definition $S(-1,-1) = 1$, $S(j,-1) = 0$ $(j \geqslant 0)$ imply

$$\frac{s^k}{\prod_{\mu=0}^{k-1}(1+\mu s)} = (-1)^k \sum_{j=k}^{\infty} S(j-1, k-1)(-s)^j.$$

Hence Proposition 3.5 becomes

$$\int_0^1 P_n^{\langle s\rangle} f(x)\,dx = \sum_{v=0}^{n} f\left(\frac{v}{n}\right)\left[\frac{1-s}{n+1} - \sum_{k=0}^{n} (-1)^k \gamma_{k+1} \left((-1)^v \binom{k}{v}\right.\right.$$
$$\left.\left. + (-1)^{n-v}\binom{k}{n-v}\right) n^{(k)} \sum_{j=k}^{\infty} S(j-1, k-1)(-s)^{j+1}\right].$$

Immediately we obtain the theorem from Definition 2.1. ∎

Now we prepare the following two lemmas for the proof of Theorem 2.5.

Lemma 4.1. *Let $n, k, v \in \mathbf{N}_0$. If $0 \leqslant k \leqslant n$ and $0 \leqslant v \leqslant n/2$ then*

$$\binom{k}{v} \geqslant \binom{k}{n-v}.$$

The proof is clearly given by the inequality

$$\binom{k}{v} - \binom{k}{n-v} = \frac{k^{(v)}}{(n-v)!}\left((n-v)^{(n-2v)} - (k-v)^{(n-2v)}\right) \geqslant 0.$$

Lemma 4.2. *For every $n \in \mathbf{N}$ and $\alpha \in \mathbf{N}_0$, if ${}_\alpha I_n$ is not positive then none of the ${}_\beta I_n$ ($\beta > \alpha$, $\beta \in \mathbf{N}_0 \cup \{\infty\}$) is positive.*

Proof. Suppose ${}_\alpha I_n$ is not positive. Then it is obvious that $\alpha \geqslant 1$ because ${}_0 I_n$ is positive. Moreover, there exists some v such that ${}_\alpha w_{n,v} < 0$, and we can assume $v \leqslant n/2$ without loss of generality because ${}_\alpha w_{n,n-v} = {}_\alpha w_{n,v}$.
Lemma 4.1 shows

$$\binom{k}{v} + (-1)^n \binom{k}{n-v} \geqslant 0 \qquad (0 \leqslant k \leqslant n).$$

Furthermore, for every $k \geqslant 0$,

$$(-1)^k \gamma_{k+1} = \frac{1}{(k+1)!}\int_0^1 x \prod_{\mu=1}^{k} (\mu - x)\,dx > 0.$$

Moreover, all the Stirling numbers of the second kind are nonnegative, as is well known. Therefore, Theorem 2.4 implies that

$${}_\alpha w_{n,v} \geqslant {}_{\alpha+1} w_{n,v} \quad \text{for all} \quad \alpha \geqslant 1 \quad \text{if} \quad v \text{ is even};$$

$${}_\alpha w_{n,v} \leqslant {}_{\alpha+1} w_{n,v} \quad \text{for all} \quad \alpha \geqslant 1 \quad \text{if} \quad v \text{ is odd}.$$

Thus ${}_1w_{n,\nu}>0$ and ${}_\alpha w_{n,\nu}<0$ imply that ν must be even and consequently

$${}_\beta w_{n,\nu} \leqslant \cdots \leqslant {}_{\alpha+1}w_{n,\nu} \leqslant {}_\alpha w_{n,\nu} < 0.$$

(Note that this inequality is valid even if $\beta=\infty$.) Therefore ${}_\beta I_n$ is not positive. ▮

Proof of Theorem 2.5. When $1 \leqslant n \leqslant 7$ or $n=9$, since all the weights in the Newton–Cotes rule are positive (see [1, p. 886] or calculate them using (3.10)), ${}_\infty I_n$ is positive. Thus $\alpha(n)=\infty$.

When $n=8$ or $n \geqslant 10$, ${}_\infty I_n$ is not positive [5, p. 156], and it must be possible to find such an $\alpha \in \mathbf{N}_0$ that ${}_\alpha I_n$ is positive but ${}_{\alpha+1}I_n$ is not positive. Then Lemma 4.2 guarantees that the found α is nothing but $\alpha(n)$.

Furthermore, for all $n \in \mathbf{N}$, Lemma 4.2 guarantees that all the ${}_\alpha I_n$ ($\alpha<\alpha(n)$, $\alpha \in \mathbf{N}_0$) are positive.

To complete the proof, we only have to determine $\alpha(n)$ for each n. We have obtained all the necessary data with the computer algebra system *Mathematica*. Here we list some of them; details are available from the author. (We let ${}_\alpha W_{n,\nu}=n\,{}_\alpha w_{n,\nu}$ in the list. Since ${}_\alpha W_{n,n-\nu}={}_\alpha W_{n,\nu}$ it suffices to examine ${}_\alpha W_{n,\nu}$ only in the case $0 \leqslant \nu \leqslant [n/2]$.)

$$\begin{aligned}
({}_{13}W_{8,\nu})_{\nu=0}^{4} &= (206412613269, 1027581059398, 212716917982,\\
&\quad 1258456018206, 87224921170)/687194767360,\\
{}_{14}W_{8,4} &= -1200668453/206158430208.\\
({}_{13}W_{10,\nu})_{\nu=0}^{5} &= (3529060133449, 18442747661870, 2368669823883,\\
&\quad 23813383005996, 117973634934,\\
&\quad 23456331479736)/12000000000000,\\
{}_{14}W_{10,4} &= -4318979540557/20000000000000.\\
({}_{13}W_{11,\nu})_{\nu=0}^{5} &= (998147479495, 5327297957586, 478794941918,\\
&\quad 7017006075333, 414657467478,\\
&\quad 4594666338516)/3423740047332,\\
{}_{14}W_{11,4} &= -383569519901/12553713506884.
\end{aligned}$$

Similarly we can obtain the necessary data for every $n \leqslant 104$.

When $n \geqslant 105$, it is of course impossible to calculate all the data, which are infinite. However, there are general and skillful expressions of them, which clearly indicate the signs of the weights. Letting $a=1/n(>0)$ and $b=1/105-1/n(\geqslant 0)$, we present them as follows:

$$({}_8W_{n,v})_{v=0}^{7}$$

$$= \Bigg(\frac{1070017}{3628800} + \frac{22418548323091}{114354828000000}\,a + \frac{25359736267}{544546800000}\,ab$$

$$+ \frac{5969355437}{93350880000}\,a^2b + \frac{12887747}{444528000}\,a^2b^2 + \frac{6151}{529200}\,a^3b^2 + \frac{1}{210}\,a^3b^3,$$

$$\frac{638662806978248}{422130126796875} + \frac{57436491956239}{42883060500000}\,b + \frac{1833683798503}{980184240000}\,b^2$$

$$+ \frac{27475395977}{18670176000}\,b^3 + \frac{3127027}{4939200}\,b^4 + \frac{289943}{2116800}\,b^5 + \frac{23}{1680}\,b^6,$$

$$\frac{103613}{403200} + \frac{4145056375238327}{1029193452000000}\,a + \frac{33409341839701}{3267280800000}\,ab$$

$$+ \frac{1395652211053}{93350880000}\,ab^2 + \frac{5753926289}{444528000}\,ab^3 + \frac{546353}{88200}\,ab^4 + \frac{3193}{2520}\,ab^5,$$

$$\frac{976909501271513}{562840169062500} + \frac{30521491285739}{4764784500000}\,b + \frac{23553221988539}{980184240000}\,b^2$$

$$+ \frac{926729577581}{18670176000}\,b^3 + \frac{875979869}{14817600}\,b^4 + \frac{80715931}{2116800}\,b^5 + \frac{51881}{5040}\,b^6,$$

$$\frac{298951}{725760} + \frac{1351609746105581}{205838690400000}\,a + \frac{62380941727861}{1960368480000}\,ab$$

$$+ \frac{1551304931951}{18670176000}\,ab^2 + \frac{10858579523}{88905600}\,ab^3 + \frac{2511179}{26460}\,ab^4 + \frac{3785}{126}\,ab^5,$$

$$\frac{419377838606459}{337704101437500} + \frac{472640006070649}{128649181500000}\,b + \frac{22235854586693}{980184240000}\,b^2$$

$$+ \frac{151076675203}{2074464000}\,b^3 + \frac{5677877201}{44452800}\,b^4 + \frac{48719261}{423360}\,b^5 + \frac{206453}{5040}\,b^6,$$

$$\frac{3349879}{3628800} + \frac{88142235701683}{68612896800000}\,a + \frac{90175372176901}{9801842400000}\,ab$$

$$+ \frac{3193508098591}{93350880000}\,ab^2 + \frac{10144953041}{148176000}\,ab^3 + \frac{36465791}{529200}\,ab^4 + \frac{67273}{2520}\,ab^5,$$

$$\frac{113426694758929}{112568033812500} + \frac{21499631502649}{128649181500000}\,b + \frac{473440190611}{326728080000}\,b^2$$

$$+ \frac{117865768367}{18670176000}\,b^3 + \frac{640930781}{44452800}\,b^4 + \frac{2274851}{141120}\,b^5 + \frac{33953}{5040}\,b^6\Bigg),$$

$${}_8W_{n,\nu}=1 \qquad (8\leqslant \nu\leqslant [n/2]),$$

$${}_9W_{n,4}$$

$$=-\left(\frac{7712181462239}{6331951901953125}+\frac{1666745232581279}{12068607656250}\,b+\frac{379336221741227}{4594613625000}\,b^2\right.$$

$$+\frac{3232816157663}{11668860000}\,b^3+\frac{375083492797}{666792000}\,b^4+\frac{21824429909}{31752000}\,b^5$$

$$\left.+\frac{35095393}{75600}\,b^6+\frac{31717}{240}\,b^7\right). \quad \blacksquare$$

5. COMPARISON BETWEEN THE TWO KINDS OF OPERATORS

This final section is devoted to comparing our new operator with Sablonnière's.

Equations (4.1) and (4.2) being applied, the Lagrange, the Sablonnière, and our operators can be represented as

$$L_nf(x)=\sum_{k=0}^{n}(B_nf)^{[k]}(x)\sum_{j=0}^{\infty}\frac{\Upsilon_{j,k}(x)}{n^j} \tag{5.1}$$

$$=\sum_{j=0}^{\infty}\frac{1}{n^j}\sum_{k=0}^{n}\Upsilon_{j,k}(x)(B_nf)^{[k]}(x)$$

$$=\sum_{j=0}^{\infty}\frac{1}{n^j}\sum_{k=0}^{2j}\Upsilon_{j,k}(x)(B_nf)^{[k]}(x), \tag{5.2}$$

$$B_n^{(K)}f(x)=\sum_{k=0}^{\{K,n\}}(B_nf)^{[k]}(x)\sum_{j=0}^{\infty}\frac{\Upsilon_{j,k}(x)}{n^j},$$

$${}_\alpha B_nf(x)=\sum_{j=0}^{\alpha}\frac{1}{n^j}\sum_{k=0}^{n}\Upsilon_{j,k}(x)(B_nf)^{[k]}(x)$$

$$=\sum_{j=0}^{\alpha}\frac{1}{n^j}\sum_{k=0}^{2j}\Upsilon_{j,k}(x)(B_nf)^{[k]}(x)$$

$$=\sum_{k=0}^{2\alpha}(B_nf)^{[k]}(x)\sum_{j=0}^{\alpha}\frac{\Upsilon_{j,k}(x)}{n^j}. \tag{5.3}$$

As we can see from the above formulas, $B_n^{(K)}$ is obtained by truncating at $k=K$ the first sum in (5.1). On the other hand, ${}_\alpha B_n$ is obtained by truncating at $j=\alpha$ the first sum in (5.2). Therefore both can be regarded as truncated operators of L_n, but the modes of truncation are distinct. It is interesting that, as (5.3) shows, ${}_\alpha B_n$ is truncated also at $k=2\alpha$ automatically, not compulsorily.

Now we itemize the advantages of our operator as follows.

(1) The value of its norm is much smaller.

(2) The parameter α corresponds exactly to the order of its convergence rate.

(3) It is simply defined with Stancu's operator and we can derive formulas about our operator from ones about Stancu's operator.

Let us compare Table II with Table I, remarking that the convergence rates of $\{{}_\alpha B_n f\}_n$, $\{B_n^{(2\alpha)} f\}_n$, and $\{B_n^{(2\alpha+1)} f\}_n$ are of the same order. Here we explain the general procedure of calculating $\|B_n^{(K)}\|$ and $\|{}_\alpha B_n\|$.

Let T be an operator represented as the form

$$Tf=\sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\tau_\nu \qquad (\tau_\nu\in\mathbf{P}_n-\{0\},\ f\colon[0,1]\to\mathbf{R})$$

and Λ be the Lebesgue function of T; that is,

$$\Lambda(x)=\sum_{\nu=0}^{n}|\tau_\nu(x)| \qquad (x\in[0,1]).$$

By solving algebraic equations numerically with computer, we can determine

$$X_0=\bigcup_{\nu=0}^{n}\{x\in(0,1)\mid \tau_\nu(x)=0\},$$

and furthermore,

$$X=X_0\cup\{0,1\}\cup\{x\in(0,1)-X_0\mid \Lambda'(x)=0\}.$$

Then

$$\|T\|=\max_{x\in[0,1]}\Lambda(x)=\max_{x\in X}\Lambda(x).$$

Tables I and II were obtained by applying the above procedure to the cases $T=B_n^{(K)}$ and $T={}_\alpha B_n$, respectively.

The two tables strongly suggest that

$$\|{}_\alpha B_n\| \leqslant \|B_n^{(2\alpha)}\| \leqslant \|B_n^{(2\alpha+1)}\| \qquad \text{for all} \quad n \in \mathbf{N} \quad \text{and} \quad \alpha \in \mathbf{N}_0.$$

This inequality is very difficult to prove and open so far, but we can explain its plausibility qualitatively, not quantitatively. Recall that $B_n^{(K)}$ ($K = 2\alpha, 2\alpha+1$) is obtained by putting $s = -1/n$ in $\sum_{k=0}^{\{K,n\}} U_k^{\langle s\rangle}(B_n f)^{[k]}$, which has poles at $s = -1/\mu$ ($\mu = 1, 2, ..., \min\{K, n\} - 1$). (See the recursion formula for $U_k^{\langle s\rangle}$ in Proposition 3.3.) We consider that these poles cause $B_n^{(K)}$ to be unstable. Therefore we should remove the poles for the sake of stability, while preserving the good convergence property of $\{B_n^{(K)}\}_n$. Fortunately, this is in fact possible by truncating at degree α the Maclaurin series of $P_n^{\langle s\rangle} f$ (regarded as a function of s). Furthermore, in this case, the number of the terms in the truncated series is minimum in order that the condition (c) in Theorem 2.4 in [3] is satisfied when we let $s = -1/n$. Consequently, we infer that ${}_\alpha B_n$ is more stable than $B_n^{(2\alpha)}$, which means the advantage (1). At the same time, we also speculate that $B_n^{(2\alpha+1)}$ is more unstable than $B_n^{(2\alpha)}$ (when $2\alpha + 1 \leqslant n$) because $\sum_{k=0}^{2\alpha+1} U_k^{\langle s\rangle}(B_n f)^{[k]}$ has the one extra pole $s = -1/(2\alpha)$ compared with $\sum_{k=0}^{2\alpha} U_k^{\langle s\rangle}(B_n f)^{[k]}$.

The advantage (2) is clear from Theorem 2.3. Furthermore, recall that the advantage (3) played an essential role in the proofs of Theorems 2.2 and 2.4. (On the other hand, Sablonnière's operator has no corresponding procedures of calculation. For example, if we want to integrate $B_n^{(K)} f$ on $[0, 1]$, we cannot help using repeated integration by parts, which is complicated.)

Therefore, we conclude that our new class of modified Bernstein operators is more natural and essential for our purpose, and more convenient both theoretically and practically, than that of Sablonnière.

In a forthcoming paper, we will present an improved result of Corollary 2.1, which weakens the differentiability condition about integrands.

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